View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Universiti Teknologi Malaysia Institutional Repository

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **26** (2003) 79–85

Parabolic-Elliptic Correspondence of a Three-Level Finite Difference Approximation to the Heat Equation

[1]MOHD SALLEH SAHIMI, [2]NORMA ALIAS, [1]NORELIZA ABU MANSOR AND [1]NORHALENA MOHD NOR

[1]Department of Engineering Sciences and Mathematics, Universiti Tenaga Nasional, 43009 Kajang, Selangor, Malaysia

[2]Department of Mathematics, Universiti Teknologi Malaysia, 81310 UTM Skudai, Johor, Malaysia

[1]e-mail: Sallehs@uniten.edu.my and [2]e-mail: norm_ally@hotmail.com

Abstract. We consider three-level difference replacements of parabolic equations focussing on the heat equation in two- space dimensions. Through a judicious splitting of the approximation, the scheme qualifies as an ADI method. Using the well-known fact of the parabolic-elliptic correspondence, we shall derive a two-stage iterative procedure employing a fractional splitting strategy applied alternately at each intermediate time step.

1. Introduction

Consider the heat equation,

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \tag{1}$$

subject to given initial and Dirichlet boundary conditions over a rectangular domain with a uniformly spaced network whose mesh points are $x_i = i\Delta x$, $t_j = j\Delta t$. If we approximate the temporal derivative by a central first difference and the space derivative by a second central difference centred at (x_i, t_j) we obtain the classic Richardson formula which is unconditionally unstable for all mesh ratios $\lambda = \frac{\Delta t}{(\Delta x)^2}$. However, a replacement of the space derivative by the average of second central differences centred at (x_i, t_{j-1}), (x_i, t_j) and (x_i, t_{j+1}) leads to the following stable and (2,2) accurate method,

$$\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t} = \frac{1}{3}\delta_x^2\left(\frac{u_{i,j+1} + u_{i,j} + u_{i,j-1}}{(\Delta x)^2}\right) \tag{2}$$

or $$\left(1-\frac{2}{3}\lambda\delta_x^2\right)u_{i,j+1}=\frac{2}{3}\lambda\delta_x^2 u_{i,j}+\left(1+\frac{2}{3}\lambda\delta_x^2\right)u_{i,j-1} \quad (3)$$

where δ is the central difference operator defined by $\delta u_i = u_{i+1/2} - u_{i-1/2}$.
As all derivatives can be expressed as an infinite series of differences, adding the next fourth-order term to (2) gives us the following stable, $(4, 2)$ accurate difference formula,

$$\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta t}=\frac{1}{3}\delta_x^2\left(\frac{u_{i,j+1}+u_{i,j}+u_{i,j-1}}{(\Delta x)^2}\right)-\frac{(\Delta x)^2\delta_t^2}{12(\Delta t)^2}u_{i,j} \quad (4)$$

The two-dimensional equivalent of (3) and (4) to approximate the heat equation in two-space,

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2} \quad (5)$$

is given respectively by,

$$\left(1-\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2\right)\right)u_{i,j,k+1}=\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2\right)u_{i,j,k}+\left(1+\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2\right)\right)u_{i,j,k-1} \quad (6)$$

and

$$\left(1-\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2\right)\right)u_{i,j,k+1}=\left(\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2\right)+\frac{4}{9}\lambda^2\alpha\delta_x^2\delta_y^2\right)u_{i,j,k}$$
$$+\left(1+\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2\right)+\frac{4}{9}\lambda^2\beta\delta_x^2\delta_y^2\right)u_{i,j,k-1} \quad (7)$$

The difference scheme (7) can be split as follows to qualify as an alternating direction implicit (ADI) scheme,

$$\left(1-\frac{2}{3}\lambda\delta_x^2\right)u_{i,j,k+1/2}=\frac{2}{3}\lambda\delta_y^2\left(\alpha u_{i,j,k}+\beta u_{i,j,k-1}+\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2\right)u_{i,j,k}\right.$$
$$\left.+\left(1+\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2\right)\right)\right)u_{i,j,k-1} \quad (8)$$

and

$$\left(1-\frac{2}{3}\lambda\delta_y^2\right)u_{i,j,k+1}=u_{i,j,k+1/2}-\frac{2}{3}\lambda\delta_y^2(\alpha u_{i,j,k}+\beta u_{i,j,k-1}) \quad (9)$$

where $\alpha+\beta=1$ and $\lambda=\dfrac{\Delta t}{(\Delta x)^2}=\dfrac{\Delta t}{(\Delta y)^2}$.

2. Other difference replacement and parabolic-elliptic correspondence

Another stable and $(4,2)$ accurate difference replacement of (5) is

$$\left(1+\left(\frac{1}{12}-\frac{2}{3}\lambda\right)\delta_x^2\right)\left(1+\left(\frac{1}{12}-\frac{2}{3}\lambda\right)\delta_y^2\right)u_{i,j,k+1}=\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2+\frac{1}{2}\delta_x^2\delta_y^2\right)u_{i,j,k}$$
$$+\left(1+\left(\frac{1}{12}+\frac{2}{3}\lambda\right)\left(\delta_x^2+\delta_y^2\right)\right)u_{i,j,k-1}+\left(\frac{1}{12}-\frac{2}{3}\lambda\right)^2\delta_x^2\delta_y^2\left(2u_{i,j,k}-u_{i,j,k-1}\right) \quad (10)$$

whose ADI analogue is given by,

$$\left(1+\left(\frac{1}{12}-\frac{2}{3}\lambda\right)\delta_x^2\right)u_{i,j,k+1/2}=-\left(\frac{1}{12}-\frac{2}{3}\lambda\right)\delta_y^2\left(2u_{i,j,k}-u_{i,j,k-1}\right)$$
$$+\frac{2}{3}\lambda\left(\delta_x^2+\delta_y^2+\frac{1}{2}\delta_x^2\delta_y^2\right)u_{i,j,k}+\left(1+\left(\frac{1}{12}+\frac{2}{3}\lambda\right)\left(\delta_x^2+\delta_y^2\right)\right)u_{i,j,k-1} \quad (11)$$

and

$$\left(1+\left(\frac{1}{12}-\frac{2}{3}\lambda\right)\delta_y^2\right)u_{i,j,k+1}=u_{i,j,k+1/2}+\left(\frac{1}{12}-\frac{2}{3}\lambda\right)\delta_y^2\left(2u_{i,j,k}-u_{i,j,k-1}\right) \quad (12)$$

As the temperature reaches steady state over time, $U\to$ constant and $\frac{\partial U}{\partial t}\to 0$ and the parabolic equation (5) reduces to the elliptic equation (Laplace's equation),

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0 \quad (13)$$

whose numerical solution can be obtained iteratively using the same ADI technique,

$$\left(1+\left(\frac{1}{12}-\frac{2}{3}r\right)\delta_x^2\right)u_{i,j}^{*}=\left(-\left(\frac{1}{12}-\frac{2}{3}r\right)\delta_y^2+\frac{2}{3}r\left(\delta_x^2+\delta_y^2+\frac{1}{2}\delta_x^2\delta_y^2\right)\right.$$
$$\left.+\left(1+\left(\frac{1}{12}+\frac{2}{3}r\right)(\delta_x^2+\delta_y^2)\right)\right)u_{i,j}^{(p)} \quad (14)$$

and

$$\left(1+\left(\frac{1}{12}-\frac{2}{3}r\right)\delta_y^2\right)u_{i,j}^{(p+1)}=u_{i,j}^{*}+\left(\frac{1}{12}-\frac{2}{3}r\right)\delta_y^2 u_{i,j}^{(p)} \quad (15)$$

where r is the acceleration parameter.

Note from the composite formula (10) that the iterative procedure converges if

$$u_{i,j,k-1} = u_{i,j,k} = u_{i,j,k+1} = u_{i,j}$$

for k sufficiently large leading to

$$\left(\delta_x^2 + \delta_y^2 + \frac{1}{6}\delta_x^2\delta_y^2\right)u_{i,j} = 0$$

which represents a nine-point difference replacement of the Laplace's equation (13). Hence we are motivated by this well-known fact of the parabolic-elliptic correspondence $[3,4]$ to develop a new iterative scheme for the solution of (1) by considering the following two-step iterates corresponding to (14) and (15),

$$(\mathbf{I} + \alpha\mathbf{G}_1)\mathbf{u}^{(p+1/2)} = \left(-\alpha\mathbf{G}_2 + \hat{\omega}\left(\mathbf{G}_1 + \mathbf{G}_2 + \frac{1}{2}\mathbf{G}_1\mathbf{G}_2\right) + \mathbf{I} + \omega(\mathbf{G}_1 + \mathbf{G}_2)\right)\mathbf{u}^{(p)} - 2r\mathbf{f} \quad (16)$$

and

$$(\mathbf{I} + \alpha\mathbf{G}_2)\mathbf{u}^{(p+1)} = \mathrm{u}^{(p+1/2)} + \alpha\mathbf{G}_2\mathbf{u}^{(p)} \quad (17)$$

where $\alpha = \frac{1}{12} - \frac{2}{3}r$, $\omega = \frac{1}{12} + \frac{2}{3}r$, $\hat{\omega} = \frac{2}{3}r$ with $r > 0$ being the acceleration parameter of the iterative process.

Noting that $\mathbf{u}^{(p+1)} = \mathbf{u}^{(p)}$ as $p \to \infty$, we have, at the $(p+1/2)$ iterate,

$$(\mathbf{I} + \alpha\mathbf{G}_1)\mathbf{u}^{(p+1/2)} = (\mathbf{I} + (\alpha + 2r)\mathbf{G}_1)(\mathbf{I} + 2r\mathbf{G}_2) + \beta\mathbf{G}_1\mathbf{G}_2)\mathbf{u}^{(p)} - 2r\mathbf{f} \quad (18)$$

and at the $(p+1)$ iterate,

$$(\mathbf{I} + \alpha\mathbf{G}_2)\mathbf{u}^{(p+1)} = \mathbf{u}^{(p+1/2)} + \alpha\mathbf{G}_2\mathbf{u}^{(p)} \quad (19)$$

where $\beta = \dfrac{2r(3\alpha - 2r)}{3}$,

and $\mathbf{G}_1$ and $\mathbf{G}_2$ are lower and upper bidiagonal matrices given by,

$$\mathbf{G}_1 = \begin{bmatrix} 1 & & & & & & \\ k_1 & 1 & & & \bigcirc & & \\ & k_2 & 1 & & & & \\ & & \cdot & \cdot & & & \\ & \bigcirc & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & k_{m-2} & 1 & \\ & & & & & k_{m-1} & 1 \end{bmatrix}_{(mxm)}, \tag{20}$$

$$\mathbf{G}_2 = \begin{bmatrix} e_1 & h & & & & & \\ & e_2 & h & & \bigcirc & & \\ & & e_3 & h & & & \\ & & & \cdot & \cdot & & \\ & \bigcirc & & & \cdot & \cdot & \\ & & & & & \cdot & \cdot \\ & & & & & e_{m-1} & h \\ & & & & & & e_m \end{bmatrix}_{(mxm)} \tag{21}$$

Elimination of $\mathbf{u}^{(p+1/2)}$ from (18) and (19) leads to the single composite formula,

$$(\mathbf{I} + \alpha\mathbf{G}_1)(\mathbf{I} + \alpha\mathbf{G}_2)\mathbf{u}^{(p+1)} = \left((\mathbf{I} + (\alpha + 2r)\mathbf{G}_1)(\mathbf{I} + 2r\mathbf{G}_2) + \beta\mathbf{G}_1\mathbf{G}_2\right)\mathbf{u}^{(p)} - 2r\mathbf{f}$$
$$+ \alpha(\mathbf{I} + \alpha\mathbf{G}_1)\mathbf{G}_2\mathbf{u}^{(p)}$$

As $p \to \infty$, $\mathbf{u}^{(p)}, \mathbf{u}^{(p+1)} \to \mathbf{u}$ resulting to

$$\mathbf{A} = \mathbf{G}_1 + \mathbf{G}_2 + \frac{1}{6}\mathbf{G}_1\mathbf{G}_2 \tag{22}$$

and

$$\mathbf{Au} = \mathbf{f} \tag{23}$$

A ia a tridiagonal matrix which arises from the difference method used to approximate the parabolic equation (1). For example, if the familiar weighted approximation,

$$-\lambda\theta u_{i-1,j+1} + (1 + 2\lambda\theta)u_{i,j+1} - \lambda\theta u_{i+1,j+1} = \lambda(1-\theta)u_{i-1,j}$$
$$+(1 - 2\lambda(1-\theta))u_{ij} + \lambda(1-\theta)u_{i+1,j}$$

is used with order of accuracy of $(1,2),(2,2),(2,4)$ and $(1,2)$ when $\theta = 1, \frac{1}{2}, \frac{1}{2} - \frac{1}{12\lambda}$ and 0 respectively, then its totality can be displayed in matrix form (23) as,

$$\begin{bmatrix} a & b & & & \bigcirc & & & \\ c & a & b & & & & & \\ & c & a & b & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & \bigcirc & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & c & a & b \\ & & & & & & c & a \end{bmatrix}_{(mxm)} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ \cdot \\ \cdot \\ u_{m-1} \\ u_m \end{bmatrix}_{j+1} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ \cdot \\ \cdot \\ f_{m-1} \\ f_m \end{bmatrix} \qquad (24)$$

Using (18) and (19), we have,

$$\mathbf{u}^{(p+1/2)} = (\mathbf{I} + \alpha \mathbf{G}_1)^{-1}((\mathbf{I} + (\alpha + 2r)\mathbf{G}_1)(\mathbf{I} + 2r\mathbf{G}_2) + \beta \mathbf{G}_1 \mathbf{G}_2)\mathbf{u}^{(p)} - 2r\mathbf{f}) \qquad (25)$$

and

$$\mathbf{u}^{(p+1)} = (\mathbf{I} + \alpha \mathbf{G}_2)^{-1}(\mathbf{u}^{(p+1/2)} + \alpha \mathbf{G}_2 \mathbf{u}^{(p)}) \qquad (26)$$

giving us the following computational formulae at each of the half-iterates, at the $(p+1/2)$ iterate,

$$u_i^{(p+1/2)} = \left(s_{i-1} u_{i-1}^{(p)} + v_i u_i^{(p)} + \hat{s} u_{i+1}^{(p)} - w_{i-1} u_{i-1}^{(p+1/2)} - r^* f_i\right)\big/ A, \quad i = 1, 2, \cdots, m \qquad (27)$$

with $s_0 = v_0 = w_0 = 0$ and $A = 1 + \frac{(1+8r)}{12} \neq 0$ at the $(p+1)$ iterate,

$$u_{m+1-i}^{(p+1)} = \left(u_{m+1-i}^{(p+1/2)} + d_{m+1-i}\, u_{m+1-i}^{(p)} + \bar{d} u_{m+2-i}^{(p)} - \bar{d} u_{m+2-i}^{(p+1)}\right)\big/(1 + d_{m+1-i}) \qquad (28)$$

with $d_i \neq 0$ for $i = 1, 2, \cdots, m$ and

$$r^* = -2r, \qquad h = \frac{6}{7} b, \qquad \alpha = \frac{1+8r}{12},$$

$$q = \frac{r^*(3\alpha - r^*)}{3}, \qquad P = \alpha - 2r, \qquad \hat{f} = r^* P,$$

$$g = \hat{f} + q, \qquad \hat{s} = h(r^* + f + q), \qquad \bar{d} = \alpha h,$$

$$\hat{A} = 1 + \alpha\,, \qquad e_1 = \frac{6(\alpha - 1)}{7}, \qquad k_i = \frac{6c}{6 + e_i},$$

$$e_{i+1} = \frac{6}{7}\left(a - \frac{k_i h}{6} - 1\right), \quad i = 2, 3, \cdots, m$$

$$s_{i-1} = Pk_{i-1} + gk_{i-1}e_{i-1}\,, \; i = 1, 2, \cdots, m$$

$$v_i = 1 + r^* e_i + P + g(hk_{i-1} + e_i)\,, \quad w_{i-1} = \alpha k_{i-1}\,, \quad d_i = \alpha e_i\,, \quad i = 1, 2, \cdots, m$$

The two-step iterative procedure of (27) and (28) corresponds to sweeping through the mesh involving at each iterate the solution of an explicit equation. This is continued until convergence is reached, that is when the convergence requirement $\left\|\mathbf{u}^{(p+1)} - \mathbf{u}^{(p)}\right\|_\infty \le \varepsilon$ is met where ε is the convergence criterion.

3. Conclusion

This paper renders a computational treatise of the derivation of a numerical scheme in the class of iterative and explicit two-steps methods to solve one-dimensional heat equations. As the basis of derivation is the unconditionally stable $(4, 2)$ accurate ADI scheme, this method is convergent, computationally stable and highly accurate. Some of the well-known schemes in this class are reported in [2]. However, the Alternating Group Explicit method employing the Douglas and Peaceman-Rachford variant, AGE-DR and AGE-PR is only $(2,1)$ and $(2, 2)$ accurate respectively. In a separate paper will be reported results of some sample problems involving the one-dimensional heat equation demonstrating the convergence, high accuracy and unconditional stability of the above scheme.

References

1. A.R. Mitchell and D.F. Griffiths, *The Finite Difference Method in Partial Differential Equations*, John Wiley and Sons, Chichester, 1980.
2. C.L. Tien and T.C. Chawla, *Annual Review of Numerical Fluid Mechanics and Heat Transfer* Vol. II, Hemisphere Publishing Corporation, New York, 1989.
3. R.S. Varga, *Matrix Iterative Analysis*, Prentice Hall International, 1962.
4. E.L. Wachspress, *Iterative Solution of Elliptic Systems*, Prentice Hall Inc., Englewood Cliffs, 1966.
5. N.N. Yanenko, *The Method of Fractional Steps*, Springer-Verlag, New York, 1971.